

# ROBUST ERROR ESTIMATES IN WEAK NORMS FOR ADVECTION DOMINATED TRANSPORT PROBLEMS WITH ROUGH DATA

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**Abstract.** We consider mixing problems in the form of transient convection–diffusion equations with a velocity vector field with multiscale character and rough data. We assume that the velocity field has two scales, a coarse scale with slow spatial variation, which is responsible for advective transport and a fine scale with small amplitude that contributes to the mixing. For this problem we consider the estimation of filtered error quantities for solutions computed using a finite element method with symmetric stabilization. A posteriori error estimates and a priori error estimates are derived using the multiscale decomposition of the advective velocity to improve stability. All estimates are independent both of the Péclet number and of the regularity of the exact solution.

**1. Introduction.** In spite of much progress in recent years the problem of deriving a posteriori error estimates and a priori error estimates for transient convection–diffusion equations remain in its infancy. This is most likely due to the wide range of different problems covered by the equation class. Indeed depending on the characteristics of the velocity field and the molecular diffusion the solutions may feature very different behaviour. The difficulty will depend mainly on the smoothness of data and on the Péclet number

$$\text{Pe}_L := \frac{UL}{\mu},$$

where  $U$  is a characteristic velocity,  $L$  is a lengthscale and  $\mu$  the molecular diffusion. If the variations of the transport velocity are small and the data are smooth, the solution will be smooth, with moderate Sobolev norm (at least the  $H^1$ -norm) independent of the viscosity, and the flow will be computable using a standard Galerkin method for low Péclet number flows and stabilized finite element methods for high Péclet number flows. In this case  $L^2$ -norm error estimates with an order can be established as we shall see below. Most difficult is the case of a high Péclet number and a strongly varying, or even turbulent, velocity field, transporting a concentration that is strongly fragmented and may dilute or concentrate. In this case a computation can experience strong amplification of errors due to repeated bifurcation of streamlines and an inexact representation of internal layers, due to spurious oscillations or numerical diffusion, leading to unphysical numerical mixing. This case is sometimes referred to as scalar turbulence and LES-models have been derived for the modelization of the passive scalar using filtering (see for example [18]). Such models encounter a similar Reynolds stress conundrum as the filtering of the full Navier-Stokes' equations, and hence the modeling error is very difficult to quantify. Another approach that has been attempted for this problem is heterogeneous multiscale methods (see for instance [14]) however in that case the underlying theory is based on homogenization and completely depending on a periodicity hypothesis that in most applications will not hold.

In this paper our approach is to apply a stabilized finite element method to the computation of the solution of the standard physical model, instead of a coarse grained model, the accuracy of the large scaled is measured by estimating the a regularized, or filtered error, equivalent to estimating the error in local averages of the solution.

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The combination of these two ingredients allows us to derive error estimates with an order in  $h$ , for a norm that is in a certain sense in between  $H^{-1}$  and  $L^2$ , but which contains the  $L^2$ -norm of a filtered error. These estimates are robust in the sense that they do not depend on any high order Sobolev norm of the exact solution and they only have an exponential growth depending on the maximum gradient of the coarse scale velocity field, under a certain scale separation assumption given below. This means that the filtered quantities considered are robust under diverging fine scale characteristic trajectories. In some sense we extract the coarse scales for which we have some (provable) accuracy from the computation.

The problem that we will consider takes the following form. Let  $\Omega$  be an open polygonal/polyhedral subset of  $\mathbb{R}^d$ ,  $d = 2, 3$  with boundary  $\partial\Omega$ ,  $u_0, f \in L^2(\Omega)$  and let  $\beta \in [C_0(0, T; W^{1,\infty}(\Omega))]^d$ ,  $\mu \in \mathbb{R}$  such that  $\nabla \cdot \beta = 0$ ,  $\beta \cdot n_{\partial\Omega}|_{\partial\Omega} = 0$ ,  $\mu > 0$ , then formally we may write, for  $t > 0$  find  $u \in H_0^1(\Omega)$  such that  $u(x, 0) = u_0(x)$  in  $\Omega$  and

$$\partial_t u + \beta \cdot \nabla u - \mu \Delta u = f, \quad \text{in } \Omega. \quad (1.1)$$

For the boundary conditions let  $u|_{\partial\Omega} = 0$ . We will denote the computational time interval by  $I := [0, T]$  and the space-time domain by  $Q := I \times \Omega$ . The  $L^2$ -scalar product over  $X$ , where  $X$  can be either a space domain of  $\mathbb{R}^d$  or a space-time domain, will be denoted by  $(\cdot, \cdot)_X$ , the  $L^2$ -scalar product over subsets  $X$  of  $\mathbb{R}^{d-1}$  will be denoted  $\langle \cdot, \cdot \rangle_X$ . In both cases the corresponding  $L^2$ -norm is denoted by  $\|\cdot\|_X$ .

In this paper the analysis will be restricted to velocity fields with a particular multiscale character. We will assume that  $\beta \in W^{1,\infty}(\Omega)$ , such that  $\nabla \cdot \beta = 0$  and that the problem is normalized so that  $U := \|\beta\|_{L^\infty(Q)} = 1$  and we assume that the characteristic lengthscale is given by  $L := 1$ , similarly we assume that  $T \approx L/\|\beta\|_{L^\infty(Q)}$  and hence also approximately one. Instead of making the standard assumption that  $\|\beta\|_{W^{1,\infty}(\Omega)}$  is small, we assume that there is a decomposition of the velocity field,

$$\beta = \bar{\beta} + \beta',$$

where, for all  $t$ ,  $\|\bar{\beta}\|_{W^{1,\infty}(\Omega)} \sim 1$  and  $\|\beta'\|_{L^\infty(\Omega)}^2 \sim \mu$  where  $\mu > 0$  is the molecular diffusion coefficient. This allows us to define a timescale for the flow relating to the coarse scale spatial variation and the fine scale amplitude,

$$(\tau_F)^{-1} := \frac{1}{2} \sup_{t \in I} \min(\|\bar{\beta}\|_{W^{1,\infty}(\Omega)}^{-1}, \|\beta'\|_{L^\infty(\Omega)}^2/\mu) \sim 1. \quad (1.2)$$

Essentially we assume that the velocity vectorfield can be decomposed in a coarse scale, responsible for transport, that is slowly varying in space and a fine scale, responsible for mixing, that has small amplitude but may have very strong spatial variation. Expressed in Péclet numbers this means that the coarse scale Péclet number may be arbitrarily high, whereas the fine scale Péclet number must be  $O(1)$ . A sharper value of  $\tau_F$  given a molecular diffusion  $\mu$  and a velocity field  $\beta$  may be obtained by solving a certain minimization problem in the  $L^\infty$ -norm that will be detailed in the a priori analysis.

We will assume that the coarse scale velocity satisfies a pointwise non-penetration condition on the domain boundary,  $\bar{\beta} \cdot n_{\partial\Omega} = 0$ . Here and in the following we used the notation  $a \lesssim b$  to denote  $a \leq Cb$  with  $C$  a moderate constant independent of the mesh-parameter and the physical parameters of the problem, (except those that are assumed to be unity.) We will also use the notation  $a \sim b$  for  $a \lesssim b$  and  $b \lesssim a$ .

We will consider rough initial data or source terms,  $u_0, f \in L^2(\Omega)$ , meaning that data can have strong fluctuation. This together with the high Péclet number and the

multiscale character of the velocity field may lead to complex, low regularity solutions. More precisely solutions are smooth, due to parabolic regularity, but with very large Sobolev norms, rendering standard a priori error estimates based on approximation theory worthless. Indeed classical global estimates for stabilized finite element methods for time dependent convection–diffusion equations [12, 4, 8] yield the high Péclet number error estimate:

$$\|u - u_h\|_{\Omega} + \|\mu^{\frac{1}{2}} \nabla(u - u_h)\|_Q \leq Ch^{\frac{3}{2}}(1 + \text{Pe}_h^{-1/2})|u|_{L^2(I; H^2(\Omega))} \quad (1.3)$$

where  $u_h$  denotes the approximate solution using piecewise affine approximation. Even though  $|u|_{H^2(\Omega)}$  is huge in the presence of layers, stabilized methods are relevant for this case since one may derive localized error estimates, showing that perturbations can not spread too far upwind of crosswind in the stationary case [16, 13, 6], or spread too far across characteristics in the transient case [19]. The reason this works is that  $|u|_{H^2(\omega)}$  can be assumed to be small in a part of the domain  $\omega \subset \Omega$  provided  $u$  has no layers in an  $h^{\frac{1}{2}}$ -neighbourhood of  $\omega$ . In the estimates the bad part can be cut away using a suitably chosen weight function. This technique is not applicable in the present case, since the strong oscillations of the velocity field and the nonsmoothness of  $u_0$  and  $f$ , makes it unrealistic to assume that  $|u|_{H^2(\omega)}$  is small in any part of the domain. The aim of this paper is to show that also in this case, stabilized finite element methods produce an improved solution compared to that of standard Galerkin and that we can indeed derive error estimates for some large scale quantities defined by differential filtering.

Drawing from earlier ideas on a posteriori error estimation (see [2, 15, 3]) we propose to estimate a regularized error, or in other terms work in a norm in between  $H^{-1}$  and  $L^2$ . Indeed let the regularized error  $\tilde{e}$  be defined by the partial differential equation

$$-\mathfrak{h}\Delta\tilde{e} + \tilde{e} = e(\cdot, T), \quad (1.4)$$

with  $\tilde{e}|_{\partial\Omega} = 0$ ,  $\mathfrak{h} \in \mathbb{R}^+$ . We then prove that

$$\|\tilde{e}\|_{\mathfrak{h}} := \left( \|\mathfrak{h}^{\frac{1}{2}} \nabla \tilde{e}\|_{\Omega}^2 + \|\tilde{e}\|_{\Omega}^2 \right)^{\frac{1}{2}} \lesssim C_T \left( \frac{h}{\mathfrak{h}} \right)^{\frac{1}{2}}.$$

The constant  $C_T$  of the estimates is independent both of the Péclet number and of the regularity of the exact solution. The main contribution to  $C_T$  is a factor

$$\exp\left(\frac{T}{\tau_F}\right)$$

where  $T$  denotes the final time of the simulation. This factor limits the applicability of the analysis to time intervals of the size  $\tau_F$  and hence limits the validity of the arguments to smooth vectorfields  $\beta$  or those satisfying the scale separation case discussed above.

The parameter  $\mathfrak{h}$  can be related to a filter width  $\delta$  by  $\mathfrak{h} := \delta^2$ , or an artificial diffusivity for elliptic smoothing. In both cases we see that the hidden constant includes dimensional quantities of order unity. For  $\mathfrak{h}$  constant the results can be interpreted as  $H^{-1}$ -norm error estimates and in this norm the convergence rate  $h^{\frac{1}{2}}$  is most likely optimal.

Below we will first recall the standard  $L^2$ -norm error analysis for high Péclet number flow and see what is required of data to obtain an estimate with an order

that is fully independent of  $\mu$  (that is, we control suitable Sobolev norms of  $u$  a priori.) Then, we consider error estimation of filtered quantities and we obtain a posteriori error estimates. A priori estimates for rough solutions in the general case follow directly using standard stability estimates of the discrete solution. A similar analysis was carried out in the nonlinear case for the one dimensional Burgers' equation in [3]. In practice the a posteriori error estimates are likely the more useful, the upper bounds however show that the method always will converge.

The weak formulation of equation (1.1) writes, for  $t > 0$ , find  $u \in H_0^1(\Omega)$  such that  $u(x, 0) = u_0(x)$  and

$$(\partial_t u, v) + a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad (1.5)$$

where  $(\cdot, \cdot)$  denotes the standard  $L^2$ -product and  $a(\cdot, \cdot)$  is defined by:

$$a(u, v) := (\beta \cdot \nabla u, v) + (\mu \nabla u, \nabla v).$$

**2. Stability and regularity of the solution.** Since the constant in the estimate (1.3) depends on the  $H^2$ -norm of the solution it is important to understand what quantities it is possible to control, both in general and in the special case that robustness in  $\mu$  is required. In this section we will first show the standard global regularity estimate for parabolic problems detailing the dependence of the constant on  $\mu$ . We will then show an estimate where the constant is independent of  $\mu$ , but not of  $\tau_F$ . In both cases we will need to assume that data have some smoothness. It is important to note that this regularity will not be required of the exact solution in the error analysis for filtered quantities that will follow.

LEMMA 2.1. (*standard energy estimate*) Let  $u$  be the solution of (1.1), then there holds

$$\sup_{t \in I} \|u(t)\|_\Omega + \|\mu^{\frac{1}{2}} \nabla u\|_Q \lesssim \int_I \|f(t)\|_\Omega dt + \|u_0\|_\Omega. \quad (2.1)$$

*Proof.* We take  $v = u$  in (1.5) and then use that

$$\int_I (f, u) dt \leq \sup_{t \in I} \|u(\cdot, t)\|_\Omega \int_I \|f\|_\Omega dt$$

from which the bound on  $\sup_{t \in I} \|u(t)\|_\Omega$  follows. This bound is then used to get the  $H^1$ -bound, which is a consequence of the equation (1.5) and the equality

$$\|\mu^{\frac{1}{2}} \nabla u\|_Q^2 = \int_I a(u, u) dt.$$

□

THEOREM 2.2. Let  $u$  be the weak solution of (1.1), with  $f \in L^2(Q)$  and  $u_0 \in H_0^1(\Omega)$  assume that  $\Omega$  is a convex domain, then there holds

$$\begin{aligned} \sup_{t \in I} \|\mu^{\frac{1}{2}} \nabla u(t)\|_\Omega + \|\mu u\|_{L^2(I; H^2(\Omega))} &\lesssim \mu^{-1/2} \left( \int_I \|f(t)\|_\Omega dt + \|u_0\|_\Omega \right) \\ &\quad + \|f\|_Q + \|\mu^{\frac{1}{2}} \nabla u_0\|_\Omega. \end{aligned}$$

*Proof.* Multiplying (1.1) with  $-\mu\Delta u$ , integrating over the space time domain  $Q^* := (0, t^*) \times \Omega$ , with  $t^* < T$  and applying the Cauchy-Schwarz inequality and the arithmetic-geometric inequality gives

$$\begin{aligned} \|\mu^{\frac{1}{2}}\nabla u(t^*)\|_{\Omega}^2 + \|\mu\Delta u\|_{Q^*}^2 &\leq \|\beta\|_{L^\infty(Q^*)}^2 \mu^{-1} \|\mu^{\frac{1}{2}}\nabla u\|_{Q^*}^2 + \frac{1}{4} \|\mu\Delta u\|_{Q^*}^2 \\ &\quad + \|f\|_{Q^*}^2 + \|\mu^{\frac{1}{2}}\nabla u_0\|_{\Omega}^2. \end{aligned}$$

It follows from (2.1) that

$$\sup_{t \in I} \|\mu^{\frac{1}{2}}\nabla u(t)\|_{\Omega} \lesssim \|\beta\|_{L^\infty(Q)} \mu^{-1/2} \left( \int_I \|f(t)\|_{\Omega} dt + \|u_0\|_{\Omega} \right) + \|f\|_Q + \|\mu^{\frac{1}{2}}\nabla u_0\|_{\Omega}$$

and

$$\|\mu\Delta u\|_Q \lesssim \|\beta\|_{L^\infty(Q)} \mu^{-1/2} \left( \int_I \|f(t)\|_{\Omega} dt + \|u_0\|_{\Omega} \right) + \|f\|_Q + \|\mu^{\frac{1}{2}}\nabla u_0\|_{\Omega}.$$

We conclude using elliptic regularity, recalling that the advection velocity is normalised.  $\square$

*Remark 1.* Observe that it follows that  $\|\nabla u\|_Q \lesssim \mu^{-1/2}$ ,  $\sup_{t \in I} \|\nabla u\|_{\Omega} \lesssim \mu^{-1}$  and  $\|u\|_{L^2(I; H^2(\Omega))} \lesssim \mu^{-3/2}$ . Hence we conclude that the regularity estimates of Lemma 2.1 and Theorem 2.2 both are sensitive to the variation of the diffusivity and can only be used when  $\text{Pe}_h \lesssim 1$ . Also some regularity of the initial data is needed. Revisiting the estimate (1.3) in this context we get

$$\|\mu^{\frac{1}{2}}\nabla(u - u_h)\|_Q \lesssim \left[ \left( \frac{h}{\mu} \right)^{3/2} + \left( \frac{h}{\mu} \right) \right] \left( \int_I \|f(t)\|_{\Omega} dt + \|u_0\|_{\Omega} + \mu^{\frac{1}{2}} \|f\|_Q + \|\mu\nabla u_0\|_{\Omega} \right). \quad (2.2)$$

This estimate is clearly useless when  $\text{Pe}_h > 1$ .

We will now show that using the scale separation of the advection field and assuming some more regularity of data, all inverse powers of the diffusivity may be avoided. The price to pay is an exponential constant, which however, under our assumption on the velocity field remains moderate. It should be noted that if no assumption is made on the velocity field the physical stability properties of the problem are very poor, since sharp, characteristic layers in the velocity may result in  $O(\mu^{\frac{1}{2}})$  width layers in the solution  $u$  and corresponding growth in the  $H^1$ -norm, making global estimates independent of  $\mu$  impossible.

**THEOREM 2.3.** *Let  $u$  be the weak solution of (1.1), with  $f \in H_0^1(\Omega)$  and  $u_0 \in H_0^1(\Omega)$ . Then for  $0 < \mathfrak{h}$  there holds*

$$\begin{aligned} \sup_{t \in I} \|u(\cdot, t)\|_{\mathfrak{h}} + \|(\mathfrak{h}\mu)^{\frac{1}{2}}\Delta u\|_Q + T^{-\frac{1}{2}} \|\mathfrak{h}^{\frac{1}{2}}\nabla u\|_Q + T^{-\frac{1}{2}} \|\mathfrak{h}^{\frac{1}{2}}\partial_t u\|_Q \\ \lesssim C_T (\mathfrak{h}^{\frac{1}{2}} \|f\|_{L^2(I; H^1(\Omega))} + \int_I \|f\| dt + \|u_0\|_{\mathfrak{h}}), \quad (2.3) \end{aligned}$$

with  $C_T = e^{\left(\frac{\tau}{\tau_F}\right)}$ , where  $\tau_F$  is given by (1.2).

*Proof.* Multiply equation (1.1) by  $-\mathfrak{h}\Delta u$  and integrate over  $Q^* := [t^*, T] \times \Omega$  with  $t^* < T$ , to obtain

$$\begin{aligned} \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}}\nabla u(\cdot, t^*)\|_{\Omega}^2 - (\beta \cdot \nabla u, \mathfrak{h}\Delta u)_{Q^*} + \|(\mu\mathfrak{h})^{\frac{1}{2}}\Delta u\|_{Q^*}^2 \\ \lesssim (\nabla f, \mathfrak{h}\nabla u)_{Q^*} + \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}}\nabla u_0\|_{\Omega}^2. \end{aligned}$$

The second term in the left hand side does not have a sign and requires some further consideration. First split the velocity field in the large and the fine scale component,

$$-(\beta \cdot \nabla u, \mathfrak{h} \Delta u)_{Q^*} = -(\bar{\beta} \cdot \nabla u, \mathfrak{h} \Delta u)_{Q^*} - (\beta' \cdot \nabla u, \mathfrak{h} \Delta u)_{Q^*},$$

then integrate by parts in the term representing the large scale transport, noting that if  $t_1$  and  $t_2$  denotes the two orthonormal tangential vectors to  $\partial\Omega$ ,

$$\bar{\beta} \cdot \nabla u|_{\partial\Omega} = \underbrace{\bar{\beta} \cdot n_{\partial\Omega}}_{=0} \nabla u \cdot n_{\partial\Omega}|_{\partial\Omega} + \sum_{i=1}^2 \bar{\beta} \cdot t_i \underbrace{(\nabla u \cdot t_i)}_{=0}|_{\partial\Omega} = 0.$$

$$-(\bar{\beta} \cdot \nabla u, \mathfrak{h} \Delta u)_{Q^*} = (\nabla(\bar{\beta} \cdot \nabla u), \mathfrak{h} \nabla u)_{Q^*}$$

Note that

$$(\nabla(\bar{\beta} \cdot \nabla u), \mathfrak{h} \nabla u)_{Q^*} = \sum_{i=1}^d ((\partial_{x_i} \bar{\beta}) \cdot \nabla u, \mathfrak{h} \partial_{x_i} u)_{Q^*} + \sum_{i=1}^d (\bar{\beta} \cdot (\partial_{x_i} \nabla u), \mathfrak{h} \partial_{x_i} u)_{Q^*}. \quad (2.4)$$

For the first term of the right hand side we have

$$\sum_{i=1}^d ((\partial_{x_i} \bar{\beta}) \cdot \nabla u, \mathfrak{h} \partial_{x_i} u)_{Q^*} = ((\nabla_S \bar{\beta}) \cdot \nabla u, \mathfrak{h} \nabla u)_{Q^*}$$

where  $\nabla_S$  denotes the symmetric part of the gradient tensor. Similarly we obtain for the second part

$$\begin{aligned} \sum_{i=1}^d (\bar{\beta} \cdot (\partial_{x_i} \nabla u), \mathfrak{h} \partial_{x_i} u)_{Q^*} &= \sum_{i=1}^d \sum_{j=1}^d (\bar{\beta}_j (\partial_{x_i} \partial_{x_j} u), \mathfrak{h} \partial_{x_i} u)_{Q^*} \\ &= \sum_{i=1}^d \sum_{j=1}^d (\bar{\beta}_j (\partial_{x_j} \partial_{x_i} u), \mathfrak{h} \partial_{x_i} u)_{Q^*} = \sum_{i=1}^d (\bar{\beta} \cdot \nabla \partial_{x_i} u, \mathfrak{h} \partial_{x_i} u)_{Q^*}. \end{aligned} \quad (2.5)$$

By the divergence theorem, recalling that  $\bar{\beta} \cdot n_{\partial\Omega} = 0$ , we have

$$\sum_{i=1}^d (\bar{\beta} \cdot \nabla \partial_{x_i} u, \mathfrak{h} \partial_{x_i} u)_{Q^*} = -\frac{1}{2} \sum_{i=1}^d (\nabla \cdot \bar{\beta} \partial_{x_i} u, \mathfrak{h} \partial_{x_i} u)_{Q^*}.$$

We conclude that, with  $\mathcal{I}$  denoting the identity matrix,

$$-(\bar{\beta} \cdot \nabla u, \mathfrak{h} \Delta u)_{Q^*} = ((\nabla_S \bar{\beta} - \frac{1}{2} \nabla \cdot \bar{\beta} \mathcal{I}) \nabla u, \mathfrak{h} \nabla u)_{Q^*} \quad (2.6)$$

Observing that

$$\begin{aligned} (\beta' \cdot \nabla u, \mathfrak{h} \Delta u)_{Q^*} &\leq \|\mathfrak{h}^{\frac{1}{2}} |\beta'| \mu^{-1/2} \nabla u\|_{Q^*} \|(\mathfrak{h} \mu)^{\frac{1}{2}} \Delta u\|_{Q^*} \\ &\leq \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}} |\beta'| \mu^{-1/2} \nabla u\|_{Q^*}^2 + \frac{1}{2} \|(\mathfrak{h} \mu)^{\frac{1}{2}} \Delta u\|_{Q^*}^2 \end{aligned} \quad (2.7)$$

we have,

$$\begin{aligned} \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}} \nabla u(\cdot, t^*)\|_{\Omega}^2 + \frac{1}{2} \|(\mu \mathfrak{h})^{\frac{1}{2}} \Delta u\|_{Q^*}^2 &\leq \mathfrak{h} \int_0^{t^*} (\nabla f, \mathfrak{h} \nabla u)_{\Omega} \\ &\quad + \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}} \nabla u_0\|_{\Omega}^2 + (\Lambda(\bar{\beta}, \beta', \mu) \nabla u, \mathfrak{h} \nabla u) \end{aligned} \quad (2.8)$$

where

$$\Lambda(\bar{\beta}, \beta', \mu) := \nabla_S \bar{\beta} - \frac{1}{2} \nabla \cdot \bar{\beta} \mathcal{I} + \frac{1}{2} |\beta'|^2 \mu^{-1}.$$

Defining

$$\tilde{\tau}_F^{-1} := \frac{1}{2} \sup_{t \in I} \inf_{\substack{\bar{\beta} \in W^{1,\infty}(\Omega) \\ \bar{\beta} \cdot n = 0 \text{ on } \partial\Omega}} \sigma_p(\Lambda(\bar{\beta}, \beta', \mu)) \quad (2.9)$$

where  $\sigma_p(\cdot)$  denotes the largest positive eigenvalue of the matrix, we may write

$$\begin{aligned} \|\mathfrak{h}^{\frac{1}{2}} \nabla u(\cdot, t^*)\|_{\Omega}^2 + \|(\mu \mathfrak{h})^{\frac{1}{2}} \Delta u\|_{Q^*}^2 &\leq t^* \|\mathfrak{h}^{\frac{1}{2}} \nabla f\|_{Q^*}^2 \\ &\quad + ((t^*)^{-1} + \tilde{\tau}_F^{-1}) \|\mathfrak{h}^{\frac{1}{2}} \nabla u\|_{Q^*}^2 + \|\mathfrak{h}^{\frac{1}{2}} \nabla u_0\|_{\Omega}^2. \end{aligned} \quad (2.10)$$

The claim regarding the control of the space derivatives now follows after an application of Gronwall's lemma, yielding

$$\|\mathfrak{h}^{\frac{1}{2}} \nabla u(\cdot, t^*)\|_{\Omega}^2 + \|(\mu \mathfrak{h})^{\frac{1}{2}} \Delta u\|_{Q^*}^2 \lesssim e^{t^*/\tilde{\tau}_F} \left( \|\mathfrak{h}^{\frac{1}{2}} \nabla f\|_{Q^*}^2 + \|\mathfrak{h}^{\frac{1}{2}} \nabla u_0\|_{\Omega}^2 \right). \quad (2.11)$$

Combining this estimate with estimate (2.1) yields the claim for the first two terms in the left hand side of equation (2.3). Note that we also have

$$T^{-\frac{1}{2}} \|\mathfrak{h}^{\frac{1}{2}} \nabla u\|_Q^2 \lesssim \sup_{t \in I} \|\mathfrak{h}^{\frac{1}{2}} \nabla u(\cdot, t)\|_{\Omega}^2 \lesssim e^{T/\tilde{\tau}_F} \left( \|\mathfrak{h}^{\frac{1}{2}} \nabla f\|_Q^2 + \|\mathfrak{h}^{\frac{1}{2}} \nabla u_0\|_{\Omega}^2 \right). \quad (2.12)$$

For the control of the time derivative, simply multiply the equation with  $-\mathfrak{h} \partial_t u$  and integrate to obtain

$$\begin{aligned} \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}} \partial_t u\|_Q^2 &\leq -(\beta \cdot \nabla u, \mathfrak{h} \partial_t u)_Q + \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}} f\|_Q^2 + \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}} \nabla u_0\|_{\Omega}^2 \\ &\leq \|\beta\|_{L^\infty(Q)} \|\mathfrak{h}^{\frac{1}{2}} \nabla u\|_Q \|\mathfrak{h}^{\frac{1}{2}} \partial_t u\|_Q + \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}} f\|_Q^2 + \frac{1}{2} \|\mathfrak{h}^{\frac{1}{2}} \nabla u_0\|_{\Omega}^2 \end{aligned}$$

and we conclude using the estimate (2.12) and the observation that comparing the definitions (1.2) and (2.9) we have  $\tau_F \leq \tilde{\tau}_F$ .  $\square$

**COROLLARY 2.4.** *If in addition  $\Omega$  is convex then there holds*

$$|(\mu \mathfrak{h})^{\frac{1}{2}} u|_{L^2(I; H^2(\Omega))} \lesssim C_T (\mathfrak{h}^{\frac{1}{2}} \|f\|_{L^2(I; H^1(\Omega))} + \int_I \|f\| dt + \|u_0\|_{\mathfrak{h}}).$$

*Proof.* If  $\Omega$  is convex then elliptic regularity holds from which

$$|(\mu \mathfrak{h})^{\frac{1}{2}} u|_{L^2(I; H^2(\Omega))} \lesssim \|(\mu \mathfrak{h})^{\frac{1}{2}} \Delta u\|_Q$$

follows.  $\square$

*Remark 2.* It is straightforward to verify that under the assumption on  $\bar{\beta}$  and  $\beta'$  made in the introduction  $\tau_F \sim 1$ .

**3. Finite element discretizations.** Let  $\{\mathcal{T}_h\}$  be a family of nonoverlapping conforming, quasi uniform triangulations,  $\mathcal{T}_h := \{K\}_h$  where the triangles  $K$  have diameter  $h_K$  and that is indexed by  $h := \max h_K$ . We let the set of interior faces  $\{F\}_h$  of a triangulation  $\mathcal{T}_h$  be denoted by  $\mathcal{F}$ .

We will consider a standard finite element space of piecewise polynomial, continuous functions

$$V_h^0 := \{v_h \in H_0^1(\Omega) : v_h|_K \in P_k(K), \quad \forall K \in \mathcal{T}_h\},$$

where  $P_k(K)$  denotes the polynomials of degree less than or equal to  $k$  on  $K$ . The following inverse inequalities are known to hold on  $V_h$ ,

$$\|\nabla v_h\|_K \leq c_i h_K^{-1} \|v_h\|_K \quad (3.1)$$

and

$$\|v_h\|_{\partial K} \leq c_t h_K^{-1/2} \|v_h\|_K, \quad (3.2)$$

The standard finite element method is then obtained by restricting the weak formulation (1.5) to the discrete space  $V_h^0$ . For  $t > 0$  find  $u_h \in V_h^0$  such that  $u_h(x, 0) = \pi_h u_0(x)$  and

$$(\partial_t u_h, v_h) + a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h^0. \quad (3.3)$$

Taking  $v_h = u_h$  we immediately get the standard stability estimate

$$\sup_{t \in (0, T]} \|u_h(t)\|_\Omega + \left( \int_0^T \|\mu^{\frac{1}{2}} \nabla u_h\|_\Omega^2 dt \right)^{\frac{1}{2}} \lesssim \int_0^T \|f\|_\Omega dt + \|u_0\|_\Omega.$$

When the Péclet number is low this is a standard parabolic equation and it may be analysed using standard techniques, see Thomée [20].

However when the local Péclet number is high, the problem is a singularly perturbed parabolic problem and the stability properties of the standard Galerkin method are in general insufficient for optimal convergence. In particular in the presence of layers the whole computational domain may be polluted by spurious oscillations, but as can be seen in Figure 3.1, convergence order is lost also for smooth solutions. In this case we study a Gaussian function convected one turn in a disc. Time discretization is performed with the Crank-Nicolson method and we compare the result of the standard Galerkin method with those obtained using a symmetric stabilization method.

In this paper we will consider symmetric stabilization methods only, however at least for time constant  $\beta$  one can obtain similar results for the SUPG-method.

**3.1. Symmetric stabilization methods.** In the last ten years there has been an important development in the field of high order symmetric stabilization methods. Such methods are typically obtained by the addition of a weakly consistent, dissipative operator to the formulation. Some of the most important symmetric stabilization methods present in the literature are the subgrid viscosity method suggested by Guermond [11], the orthogonal subscale method proposed by Codina [9], the local project method introduced by Becker and Braack [1], the discontinuous Galerkin method [17] and the continuous interior penalty (CIP) method suggested by Douglas and Dupont [10] and analysed by Burman and Hansbo [7]. The analysis that we propose herein



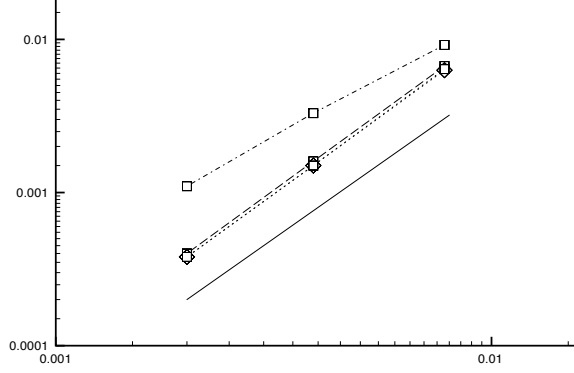


FIG. 3.1. Convergence in the  $L^2$ -norm plotted against the timestep  $\tau = h$ , of the stabilized (dashed line for explicit and dotted line for implicit treatment of the stabilization operator) and un-stabilized (dash-dot) Crank-Nicolson method. The full line shows optimal second order convergence.

will rely on an orthogonality argument and hence is valid only for the orthogonal subscales, the discontinuous Galerkin method and for the CIP-method. To reduce the notational overhead we also exclude the DG-method from the discussion, but to partisans of this method the extension of the arguments should be straightforward.

Herein we will assume that the following holds

1.  $s_h(\cdot, \cdot) : V_h \times V_h \mapsto \mathbb{R}$  is a symmetric, bilinear and positive semi definite operator.
2. For the  $L^2$ -projection  $\pi_h : L^2(\Omega) \mapsto V_h$  there holds
  - Approximability

$$\|h^{-1/2}(u - \pi_h u)\|_\Omega + \|\mu \nabla(u - \pi_h u)\|_\Omega \lesssim (h^{\frac{1}{2}} + \mu^{\frac{1}{2}})h^k |u|_{H^{k+1}(\Omega)} \quad (3.4)$$

- Weak consistency of the stabilization operator

$$s_h(\pi_h u, \pi_h u) \lesssim c_s h^{k+1/2} |u|_{H^{k+1}(\Omega)} \quad (3.5)$$

- Enhanced continuity of the convection term: For all  $v_h \in V_h$  and  $\beta \in W^{1,\infty}(\Omega)$ , with  $\beta \cdot n_{\partial\Omega}|_{\partial\Omega} = 0$ , there holds

$$\begin{aligned} |(u - \pi_h u, \beta \cdot \nabla v_h)| &\lesssim \|\beta\|_{W^{1,\infty}(\Omega)} \|u - \pi_h u\|_\Omega \|v_h\|_\Omega \\ &\quad + \|\beta\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|h^{-1/2}(u - \pi_h u)\|_\Omega s_h(v_h, v_h). \end{aligned} \quad (3.6)$$

As an example consider the CIP-method. Here  $s_h(\cdot, \cdot)$  consists in a penalty on the jump of the gradient over element faces, and takes the form

$$s_h(u_h, v_h) := \gamma \sum_K \sum_{F \in \partial K \setminus \Omega} \langle h_F^2 \|\beta \cdot n_F\|_{L^\infty(F)} \llbracket \nabla u_h \cdot n_F \rrbracket, \llbracket \nabla v_h \cdot n_F \rrbracket \rangle_F$$

where  $F$  denotes the faces in the mesh,  $\llbracket x \rrbracket$  the jump of  $x$  over  $F$ , the orientation is not important,  $n_F$  a fixed but arbitrary normal associated to each face and  $\langle \cdot, \cdot \rangle_X$  the  $L^2$ -scalar product over the  $d-1$ -dimensional domain  $X$ . In the analysis below we will use this stabilization operator.

The stabilized finite element method then takes the form, for  $t > 0$  find  $u_h \in V_h$  such that  $u_h(x, 0) = \pi_h u_0(x)$  and

$$(\partial_t u_h, v_h) + a(u_h, v_h) + s_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \quad (3.7)$$

For the satisfaction of the assumptions (3.4)–(3.6) we refer to [5]. Although in that reference Nitsche-type boundary conditions are used, the same argument works whenever  $\beta \cdot \nabla v_h|_{\partial\Omega} = 0$ , which will be the case herein. For completeness we show how to modify the result of [5] to obtain (3.6) in our case. We also show that the flow field  $\beta$  in the stabilization operator may be replaced by  $\bar{\beta}$  at the cost of a nonessential perturbation. The key result is the following Lemma.

LEMMA 3.1. *Assume that  $\beta \cdot n_{\partial\Omega}|_{\partial\Omega} = 0$  then there holds*

$$\inf_{v_h \in V_h} \|h^{\frac{1}{2}}(\pi_0 \beta \cdot \nabla u_h - v_h)\|_{\Omega} \lesssim s_h(u_h, u_h)^{\frac{1}{2}} + h^{\frac{1}{2}} \|\beta\|_{W^{1,\infty}(\Omega)} \|u_h\|_{\Omega},$$

where  $\pi_0 \beta$  is a piecewise constant approximation of  $\beta$  that will be defined below.

*Proof.* Let  $\pi_0 \beta$  be the projection onto element wise constants such that for every  $K$  that has no face on the boundary there holds

$$\int_K \pi_0 \beta \, dx = \int_K \beta \, dx.$$

On elements adjacent to the boundary define  $\pi_0 \beta$  by

$$\int_{\partial K \cup \partial\Omega} \pi_0 \beta \cdot n_{\partial\Omega} \, dx = \int_{\partial K \cup \partial\Omega} \beta \cdot n_{\partial\Omega} \, dx$$

and

$$\int_{\partial K \cup \partial\Omega} \pi_0 \beta \cdot t_i \, dx = \int_{\partial K \cup \partial\Omega} \beta \cdot t_i \, dx, \quad i = 1, \dots, d-1.$$

It follows from standard approximation that for all  $K \in \mathcal{T}_h$ ,

$$\|\beta - \pi_0 \beta\|_{L^\infty(K)} \lesssim \|\beta\|_{W^{1,\infty}(K)} h_K.$$

Note that for any element  $K$  with one face on the boundary there holds

$$\pi_0 \beta \cdot \nabla u_h|_{\partial\Omega} = \underbrace{\pi_0 \beta \cdot n_{\partial\Omega}}_{=0} \nabla u_h \cdot n_{\partial\Omega}|_{\partial\Omega} + \sum_{i=1}^2 \pi_0 \beta \cdot t_i (\underbrace{\nabla u_h \cdot t_i}_{=0})|_{\partial\Omega} = 0.$$

It then follows using the same arguments as in [5] that

$$\inf_{v_h \in V_h} \|h^{\frac{1}{2}}(\pi_0 \beta \cdot \nabla u_h - v_h)\|_{\Omega} \leq \left( \sum_{F \in \mathcal{F}} \|h[\pi_0 \beta \cdot \nabla u_h]\|_F^2 \right)^{\frac{1}{2}}.$$

By adding and subtracting  $\beta$  inside the jump and by applying a triangle inequality followed by a trace inequality we have

$$\left( \sum_{F \in \mathcal{F}} \|h[\pi_0 \beta \cdot \nabla u_h]\|_F^2 \right)^{\frac{1}{2}} \lesssim s_h(u_h, u_h)^{\frac{1}{2}} + h^{\frac{1}{2}} \|\beta\|_{W^{1,\infty}(\Omega)} \|u_h\|_{\Omega}$$

and the proof is finished.  $\square$

The continuity (3.6) is now obtained by adding and subtracting  $\pi_0\beta$  in the right slot of the left hand side of the equation

$$\begin{aligned} |(u - \pi_h u, \beta \cdot \nabla v_h)| &\leq |(u - \pi_h u, (\beta - \pi_0 \beta) \cdot \nabla v_h)| + \inf_{w_h \in V_h} |(u - \pi_h u, \pi_0 \beta \cdot \nabla v_h - w_h)| \\ &\lesssim \|\beta\|_{W^{1,\infty}(\Omega)} \|u - \pi_h u\|_{\Omega} \|v_h\|_{\Omega} \\ &\quad + \|\beta\|_{L^\infty(\Omega)} \|h^{-1/2}(u - \pi_h u)\|_{\Omega} s_h(v_h, v_h). \end{aligned} \quad (3.8)$$

where a Cauchy-Schwarz inequality, an inverse inequality, the approximation properties of  $\pi_0\beta$  and Lemma 3.1 have been used in the last inequality.

LEMMA 3.2. *Let*

$$\bar{s}_h(u_h, v_h) := \gamma \sum_K \sum_{F \in \mathcal{F}} \langle h_F^2 \|\bar{\beta} \cdot n_F\|_{L^\infty(F)} [\nabla u_h \cdot n_F], [\nabla v_h \cdot n_F] \rangle_F.$$

Then

$$s_h(u_h, u_h) \lesssim \bar{s}_h(u_h, u_h) + h^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \nabla u_h\|_{\Omega}^2$$

and

$$\bar{s}_h(u_h, u_h) \lesssim s_h(u_h, u_h) + h^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \nabla u_h\|_{\Omega}^2.$$

*Proof.* The proof for the upper and the lower bounds are similar, so we consider only the first inequality. By using the decomposition of  $\beta$  and a triangular inequality we have

$$s_h(u_h, u_h) \lesssim \gamma \sum_{F \in \mathcal{F}} \|h_F \|\bar{\beta} \cdot n_F\|_{L^\infty(F)} [\nabla u_h]_F^2 + \gamma \sum_{F \in \mathcal{F}} \|h_F \|\beta' \cdot n_F\|_{L^\infty(\Omega)} [\nabla u_h]_F^2.$$

Using now the size constraint on  $\beta'$  and a trace inequality we conclude

$$\gamma \sum_{F \in \mathcal{F}} \|h_F \|\beta' \cdot n_F\|_{L^\infty(\Omega)} [\nabla u_h]_F^2 \lesssim \gamma \sum_{K \in \mathcal{T}_h} \|h_K^{\frac{1}{2}} \mu^{\frac{1}{2}} \nabla u_h\|_K^2 \lesssim \gamma h^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \nabla u_h\|_{\Omega}^2.$$

$\square$

**3.1.1. Stability and convergence results.** We will now recall the main results on stability and convergence of symmetric stabilization methods. These results are minor modifications of those in [4]. For convenience we introduce a triple norm associated to the stabilized method. Let

$$|||u_h|||_h^2 := \int_I \left( \|\mu^{\frac{1}{2}} \nabla u_h\|^2 + s_h(u_h, u_h) \right) dt.$$

Following the proof of Lemma 2.1 it is straightforward to derive the following stability and error estimates.

LEMMA 3.3. (*Stability*)

Let  $u_h$  be the solution of (3.7), with  $\gamma > 0$  and  $\gamma_{bc} \geq c_0 > 0$ ,  $c_0$  large enough, then there holds

$$\sup_{t \in I} \|u_h(t)\|_{\Omega} + |||u_h|||_h \lesssim \int_0^T \|f\|_{\Omega} dt + \|u_0\|_{\Omega}.$$

Applying stability to the error equation leads to the following error estimate:

**THEOREM 3.4.** (*Convergence CIP-method*)

Let  $u \in L^2(I; H^r(\Omega))$  be the solution of (1.5) and  $u_h$  the solution of (3.7). Then there holds

$$\sup_{t \in I} \|u - u_h(t)\|_\Omega + \|u - u_h\|_h \lesssim (\tau_F^{-1} T^{\frac{1}{2}} h + (\tau_F^{-\frac{1}{2}} h^{\frac{1}{2}} + 1) h^{\frac{1}{2}} + \mu^{\frac{1}{2}}) h^{s-1} \|u\|_{L^2(I; H^s(\Omega))},$$

with  $s = \min(r, k+1)$  and where we used that  $\|\beta\|_{L^\infty(Q)} \sim 1$ .

*Proof.* For simplicity we only give the proof in the form of a final time result. Let  $e_h := u_h - \pi_h u$ , using the same arguments as for the stability Lemma 2.1 we have

$$\frac{1}{2} \|e_h(T)\|_\Omega^2 + \|e_h\|_h^2 = \int_I [(\partial_t e_h, e_h)_\Omega + a(e_h, e_h) + s_h(e_h, e_h)] \, dt.$$

Using Galerkin orthogonality and the orthogonality of the  $L^2$ -projection we have

$$\frac{1}{2} \|e_h(T)\|_\Omega^2 + \|e_h\|_h^2 = \int_I [a(u - \pi_h u, e_h) + s_h(\pi_h u, e_h)] \, dt.$$

Using the decomposition of the velocity we have

$$a(u - \pi_h u, e_h) \lesssim (u - \pi_h u, \bar{\beta} \cdot \nabla e_h) + (u - \pi_h u, \beta' \cdot \nabla e_h) + \|\mu^{\frac{1}{2}} \nabla(u - \pi_h u)\|_\Omega \|\mu^{\frac{1}{2}} \nabla e_h\|_\Omega$$

and using the continuity (3.6) in the first term of the right hand side and the bound on  $\beta'$  in the second we have

$$\begin{aligned} a(u - \pi_h u, e_h) &\lesssim \|\nabla \bar{\beta}\|_{L^\infty(\Omega)} \|(u - \pi_h u)\|_\Omega \|e_h\|_\Omega \\ &+ (\|\bar{\beta}\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|h^{-1/2}(u - \pi_h u)\|_\Omega + \tau_F^{-\frac{1}{2}} \|u - \pi_h u\|_\Omega + \|\mu^{\frac{1}{2}} \nabla(u - \pi_h u)\|_\Omega) \|e_h\|_h. \end{aligned}$$

It follows after a Cauchy-Schwarz inequality and an arithmetic-geometric inequality in the right hand side that

$$\begin{aligned} \|e_h(T)\|_\Omega^2 + \|e_h\|_h^2 &\lesssim \|\nabla \bar{\beta}\|_{L^\infty(Q)}^2 T \|h^{-1/2}(u - \pi_h u)\|_Q^2 \\ &+ (\tau_F^{-1} + \|\beta\|_{L^\infty(Q)} h^{-1}) \|u - \pi_h u\|_Q^2 + \|\mu^{\frac{1}{2}} \nabla(u - \pi_h u)\|_Q^2 + T^{-1} \|e_h\|_Q^2. \end{aligned}$$

We conclude by applying Gronwall's lemma and the approximation results of (3.4).  $\square$  For high mesh Péclet numbers this estimate is optimal in the  $L^\infty(I; L^2(\Omega))$ -norm and for low mesh Péclet numbers it is optimal in the  $L^2(I; H^1(\Omega))$ -norm. In the latter case the convergence in the  $L^2$ -norm can be improved under certain assumptions on the time variation of  $\beta$ , see [20]. Note that this estimate does not have exponential growth, however that factor is hidden in the Sobolev norm of the exact solution as we shall see. Combining this convergence result with the regularity result of Theorem 2.3 we may prove the following estimate that is fully independent of  $\mu$ , in the sense that we also control the Sobolev norm in the constant. Note however that smoothness of the source term and the initial data is required.

**COROLLARY 3.5.** Let  $f \in L^2(I; H_0^1(\Omega))$  and  $u_0 \in H_0^1(\Omega)$ , let  $u$  denote the solution of (1.5) and  $u_h$  the solution of (3.7) and assume that  $\text{Pe}_h \gg 1$ . Then

$$\|(u - u_h)(T)\|_\Omega \lesssim C_T (T^{\frac{1}{2}} (1 + \tau_F^{-1}) + 1) h^{\frac{1}{2}} (\|f\|_{L^2(I; H^1(\Omega))} + \|\nabla u_0\|_\Omega).$$

*Proof.* An immediate consequence of the Theorem 3.4 and Theorem 2.3.  $\square$

**4. Perturbation equation and the dual problem.** The analysis uses a perturbation equation and an associated dual problem. In this work we use the perturbation equation associated to the weak form of the equations (1.5) and the standard Galerkin formulation (3.3), for the perturbation equation. Taking the difference of the two formulations, setting  $e = u - u_h$  and integrating by parts we obtain

$$(\partial_t e, \varphi) + a(e, \varphi) = (e(T), \varphi(T)) - (e(0), \varphi(0)) - (e, \partial_t \varphi + \beta \cdot \nabla \varphi) + (\mu \nabla e, \nabla \varphi).$$

This suggests the adjoint equation, find  $\varphi \in H_0^1(\Omega)$  such that

$$\begin{aligned} -\partial_t \varphi - \beta \cdot \nabla \varphi - \mu \Delta \varphi &= 0 \text{ in } \Omega \\ \varphi &= 0 \text{ on } \partial\Omega \\ \varphi(\cdot, T) &= \Psi(\cdot) \text{ in } \Omega. \end{aligned} \tag{4.1}$$

So that the following error representation holds

$$(e(T), \Psi) = (e(0), \varphi(0)) + (\partial_t e, \varphi) + a(e, \varphi). \tag{4.2}$$

Observe that since  $\tilde{e}|_{\partial\Omega} = 0$  and  $\varphi|_{\partial\Omega} = 0$  we have  $\partial_t \varphi|_{\partial\Omega} = 0$ ,

$$(\beta \cdot \nabla \varphi)|_{\partial\Omega} = \underbrace{\beta \cdot n_{\partial\Omega}}_{=0} \nabla \varphi \cdot n_{\partial\Omega} + \beta \times n_{\partial\Omega} \cdot \underbrace{\nabla \varphi \times n_{\partial\Omega}}_{=0} = 0.$$

Finally, by the equation we have  $\Delta \varphi|_{\partial\Omega} = 0$ . These properties will be useful to prove the necessary bounds on the dual solution. The properties of the continuous dual problem will be crucial for our argument. We will now proceed and discuss the choice of  $\Psi$  and the associated stability estimates on  $\varphi$ .

**4.1. Regularization of the error and weak norms.** Since it appears not to be possible to prove a posteriori error estimates for the  $L^2$ -norm independently of the Péclet number, unless one resorts to a saturation assumption, we will here consider a regularized error, where a parameter  $\mathfrak{h}$  that may ultimately depend on  $h$  sets the scale of the regularization. We recall the problem (1.4) for a given computational error  $e := u - u_h$ , find  $\tilde{e}$  such that  $\tilde{e}|_{\partial\Omega} = 0$  and

$$-\mathfrak{h} \Delta \tilde{e} + \tilde{e} = e(\cdot, T).$$

On weak form the problem writes: find  $\tilde{e} \in H_0^1(\Omega)$  such that

$$(\mathfrak{h} \nabla \tilde{e}, \nabla v) + (\tilde{e}, v) = (e, v), \quad \forall v \in H_0^1(\Omega). \tag{4.3}$$

This is what is commonly called a Helmholtz filter or a differential filter, although it is not properly speaking a filter. The key observation here is that when  $\mathfrak{h}$  is small, the filtered solution is close to the solution where ever the solution is smooth. Close to layers or other strongly localised features of  $e$ ,  $\tilde{u} - u$  may be large locally, also for  $\mathfrak{h}$  small. Associated with the problem (4.3) we have the norm

$$\|\tilde{e}\|_{\mathfrak{h}}^2 := \|\mathfrak{h}^{\frac{1}{2}} \nabla \tilde{e}\|^2 + \|\tilde{e}\|^2$$

and an associated relation, obtained by testing (4.3) with  $v = \tilde{e}$ ,

$$\|\tilde{e}\|_{\mathfrak{h}}^2 = (e, \tilde{e}). \tag{4.4}$$

We deduce from (4.4) that choosing  $\Psi = \tilde{e}$  in (4.1) above leads to the following error representation for  $\|\tilde{e}\|_{\mathfrak{h}}^2$ :

$$\|\tilde{e}\|_{\mathfrak{h}}^2 = (e(0), \varphi(0)) + (\partial_t e, \varphi) + a(e, \varphi). \tag{4.5}$$

**4.2. Stability of the dual solution.** The advantage of using the dual technique is that instead of relying on regularity estimates for the exact solutions we can use regularity of the adjoint equation, which may be better behaved provided the data is well chosen. The following Theorem gives a precise characterization of the regularity of the dual problem in the multiscale framework.

**THEOREM 4.1.** *Let  $\varphi(x, t)$  be the weak solution of (4.2), with  $\Psi = \tilde{e}$  then there holds*

$$\sup_{t \in I} \|\mathfrak{h}^{\frac{1}{2}} \nabla \varphi(\cdot, t)\|_{\Omega} + T^{-1} \|\mathfrak{h}^{\frac{1}{2}} \nabla \varphi\|_Q + T^{-1} \|\mathfrak{h}^{\frac{1}{2}} \partial_t \varphi\|_Q + \|(\mathfrak{h}\mu)^{\frac{1}{2}} \Delta \varphi\|_Q \lesssim C_T \|\tilde{e}\|_{\mathfrak{h}},$$

with  $C_T = e^{\left(\frac{\tau}{\tau_F}\right)}$  where  $\tau_F$  is given by (1.2). If in addition  $\Omega$  is convex, then

$$|(\mathfrak{h}\mu)^{\frac{1}{2}} \varphi|_{L^2(I; H^2(\Omega))} \lesssim C_T \|\tilde{e}\|_{\mathfrak{h}}$$

*Proof.* The dual problem is equivalent to the forward problem after a change of variable  $\tilde{t} = -t$  and  $\tilde{x} = -x$ , with the source term  $f = 0$  and the initial data  $u_0 = \tilde{e}$ . The result then follows from Theorem 2.3 and Corollary 2.4.  $\square$

**5. Error estimates.** In this section we will prove estimates for  $\|\tilde{e}\|_{\mathfrak{h}}$  where the constant is robust in  $\mu$  (under our assumptions on the data). We will only consider the case of semi discretization in space and show how to prove a posteriori error estimates, where the stability constant is essentially  $C_T$  of Theorem 4.1. The a priori error estimates then follows using the fact that the a posteriori residuals are a priori controlled by the discrete stability estimate (3.3). We will first consider the case where no convexity assumption is made on the domain so that elliptic regularity can not be assumed. Then we will consider the case of insufficient data, i.e. when only  $\bar{\beta}$  is known, and show that under our assumptions on data we can obtain an upper bound of the error also in this case, where a nonconsistent part limits the asymptotic convergence, in the regime that we are interested in however this part is smaller than the discretization error.

In the low Péclet number regime, we need to modify our estimate to obtain optimality. These modifications are possible only if we assume that the domain is convex so that elliptic regularity holds. In that case the estimates below can be made robust also in the limit of low Péclet numbers by straightforward modifications. We omit the details.

**5.1. A posteriori and a priori error estimates.** To prove a posteriori error estimates we use a duality technique together with the a priori control of the dual solution. In practice one may need to resort to numerical solution of the dual problem. In the multiscale framework it is however unclear if the norm of the gradient of the

dual solution is computable or not in general.

**THEOREM 5.1.** *(A posteriori error estimate) Let  $\tilde{e}$  be defined by (4.3). Then there holds*

$$\begin{aligned} \|\tilde{e}\|_{\mathfrak{h}} &\lesssim C_T \left(\frac{h}{\mathfrak{h}}\right)^{\frac{1}{2}} \left( \int_I \inf_{v_h \in V_h} \|h^{\frac{1}{2}}(\beta \cdot \nabla u_h - v_h)\|_{\Omega} dt \right. \\ &\quad + \int_I \left( \inf_{v_h \in V_h} \sum_{K \in \mathcal{T}_h} \|h^{\frac{1}{2}}(\mu \Delta u_h - v_h)\|_K^2 \right)^{\frac{1}{2}} dt + \int_I \left( \sum_{F \in \mathcal{F}} \|\mu \llbracket \nabla u_h \cdot n_F \rrbracket\|_F^2 \right)^{\frac{1}{2}} dt \\ &\quad \left. + \int_I s_h(u_h, u_h)^{\frac{1}{2}} dt + h^{\frac{1}{2}} \int_I \|f - \pi_h f\|_{\Omega} dt + h^{\frac{1}{2}} \|u_0 - \pi_h u_0\|_{\Omega} \right). \quad (5.1) \end{aligned}$$

*Proof.* Starting from (4.5) and using Galerkin orthogonality and the orthogonality of the  $L^2$ -projection we have

$$\|\tilde{e}\|_{\mathfrak{h}}^2 = (e(0), \varphi(0) - \pi_h \varphi)_{\Omega} + (\partial_t e, \varphi - \pi_h \varphi)_Q + \int_I (a(e, \varphi - \pi_h \varphi) + s_h(u_h, \pi_h \varphi)) dt. \quad (5.2)$$

Using the weak formulation (1.5) and the orthogonality of the  $L^2$ -projection we may write

$$\begin{aligned} \|\tilde{e}\|_{\mathfrak{h}}^2 &= (f - \pi_h f, \pi_h \varphi - \varphi)_Q + (e(0), \varphi(0) - \pi_h \varphi(0))_{\Omega} + (\beta \cdot \nabla u_h - v_h, \pi_h \varphi - \varphi)_Q \\ &\quad + (\mu \nabla u_h, \nabla(\pi_h \varphi - \varphi))_Q + \int_I s_h(u_h, \pi_h \varphi) dt. \quad (5.3) \end{aligned}$$

Considering the right hand side term by term we get using Cauchy-Schwarz inequality, approximation and the stability of Theorem 4.1

$$\begin{aligned} (f - \pi_h f, \pi_h \varphi - \varphi)_Q &\leq h \mathfrak{h}^{-1/2} \int_I \|f - \pi_h f\|_{\Omega} dt \sup_{t \in I} \|\mathfrak{h}^{\frac{1}{2}} \nabla \varphi\|_{\Omega} \\ &\lesssim C_T \left(\frac{h}{\mathfrak{h}}\right)^{\frac{1}{2}} h^{\frac{1}{2}} \int_I \|f - \pi_h f\|_{\Omega} dt \|\tilde{e}\|_{\mathfrak{h}}, \quad (5.4) \end{aligned}$$

$$\begin{aligned} (e(0), \varphi(0) - \pi_h \varphi(0))_{\Omega} &\leq h \mathfrak{h}^{-1/2} \|e(0)\|_{\Omega} \sup_{t \in I} \|\mathfrak{h}^{\frac{1}{2}} \nabla \varphi\|_{\Omega} \\ &\lesssim C_T \left(\frac{h}{\mathfrak{h}}\right)^{\frac{1}{2}} h^{\frac{1}{2}} \|e(0)\|_{\Omega} \|\tilde{e}\|_{\mathfrak{h}} \quad (5.5) \end{aligned}$$

$$\begin{aligned} (\beta \cdot \nabla u_h - v_h, \pi_h \varphi - \varphi)_Q &\leq h^{\frac{1}{2}} \mathfrak{h}^{-1/2} \int_I \|h^{\frac{1}{2}}(\beta \cdot \nabla u_h - v_h)\|_{\Omega} dt \sup_{t \in I} \|\mathfrak{h}^{\frac{1}{2}} \nabla \varphi(\cdot, t)\|_{\Omega} \\ &\lesssim C_T \left(\frac{h}{\mathfrak{h}}\right)^{\frac{1}{2}} \int_I \|h^{\frac{1}{2}}(\beta \cdot \nabla u_h - v_h)\|_{\Omega} dt \|\tilde{e}\|_{\mathfrak{h}}. \quad (5.6) \end{aligned}$$

In the fourth term we first integrate by parts on each element and then proceed with trace inequalities, followed by approximation for the dual solution,

$$\begin{aligned}
& (\mu \nabla u_h, \nabla(\pi_h \varphi - \varphi))_Q \\
& \lesssim \underbrace{\int_I \left( \inf_{v_h \in V_h} \sum_{K \in \mathcal{T}_h} \|h^{\frac{1}{2}}(\mu \Delta u_h - v_h)\|_K^2 \right)^{\frac{1}{2}} + \left( \sum_{F \in \mathcal{F}} \|\mu \llbracket \nabla u_h \cdot n_F \rrbracket\|_F^2 \right)^{\frac{1}{2}} dt}_{\mathcal{R}(u_h, \mu)} \\
& \quad \times \sup_{t \in I} \left( \sum_{F \in \mathcal{F}} \|\pi_h \varphi - \varphi\|_F^2 + \|h^{-1/2}(\pi_h \varphi - \varphi)\|_\Omega^2 \right)^{\frac{1}{2}} \\
& \quad \lesssim \left( \frac{h}{\mathfrak{h}} \right)^{\frac{1}{2}} \mathcal{R}(u_h, \mu) \sup_{t \in I} \|\mathfrak{h}^{\frac{1}{2}} \nabla \varphi(\cdot, t)\|_\Omega \\
& \quad \lesssim C_T \left( \frac{h}{\mathfrak{h}} \right)^{\frac{1}{2}} \mathcal{R}(u_h, \mu) \|\tilde{e}\|_{\mathfrak{h}}. \quad (5.7)
\end{aligned}$$

Finally the stabilization term is handled using a Cauchy-Schwarz inequality, followed by a trace inequality and the  $H^1$ -stability of the  $L^2$ -projection on quasi-uniform meshes.

$$\begin{aligned}
\int_I s_h(u_h, \pi_h \varphi) dt & \lesssim \|\beta\|_{L^\infty(Q)}^{\frac{1}{2}} \int_I s(u_h, u_h)^{\frac{1}{2}} dt h^{\frac{1}{2}} \sup_{t \in I} \|\nabla \varphi(\cdot, t)\|_\Omega \\
& \lesssim C_T \left( \frac{h}{\mathfrak{h}} \right)^{\frac{1}{2}} \int_I s(u_h, u_h)^{\frac{1}{2}} dt \|\tilde{e}\|_{\mathfrak{h}}. \quad (5.8)
\end{aligned}$$

The claim follows by collecting the upper bounds (5.4)-(5.8) and dividing by  $\|\tilde{e}\|_{\mathfrak{h}}$ .  $\square$

**THEOREM 5.2.** (*A priori error estimate*) Assume that  $\text{Pe}_h > 1$  then there holds

$$\|\tilde{e}\|_{\mathfrak{h}} \lesssim C_T \left( \frac{h}{\mathfrak{h}} \right)^{\frac{1}{2}} (1 + h^{\frac{1}{2}} + T^{\frac{1}{2}}) \left( \int_0^T \|f\|_\Omega dt + \|u_0\|_\Omega \right).$$

*Proof.* The result follows from estimate (5.1) by bounding all the residual terms using (3.3). First observe that since the Péclet number is high we may use inverse inequalities and trace inequalities to show that

$$\mathcal{R}(u_h, \mu) \lesssim T^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \nabla u_h\|_Q \leq T^{\frac{1}{2}} \left( \int_0^T \|f\|_\Omega dt + \|u_0\|_\Omega \right).$$

For the contributions on the faces we have used

$$\begin{aligned}
\int_I \|\mu \llbracket \nabla u_h \cdot n_F \rrbracket\|_{\mathcal{F}} dt & \lesssim \|\beta\|_{L^\infty(\Omega)}^{\frac{1}{2}} h^{\frac{1}{2}} T^{\frac{1}{2}} \left( \int_I \sum_{F \in \mathcal{F}} \|\mu^{\frac{1}{2}} \llbracket \nabla u_h \cdot n_F \rrbracket\|_F^2 dt \right)^{\frac{1}{2}} \\
& \lesssim \|\beta\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \nabla u_h\|_Q \lesssim \|u_h\|_h. \quad (5.9)
\end{aligned}$$



Similarly using a Cauchy-Schwarz inequality in time we have

$$\int_T s(u_h, u_h)^{\frac{1}{2}} dt \leq T^{\frac{1}{2}} \int_T s(u_h, u_h) dt \leq T^{\frac{1}{2}} \left( \int_0^T \|f\|_{\Omega} dt + \|u_0\|_{\Omega} \right)$$

We finally consider the first term on the right hand side of (5.1). First note that

$$\int_I \inf_{v_h \in V_h} \|h^{\frac{1}{2}}(\beta \cdot \nabla u_h - v_h)\|_{\Omega} dt \leq T^{\frac{1}{2}} \inf_{v_h \in V_h} \|h^{\frac{1}{2}}(\bar{\beta} \cdot \nabla u_h - v_h)\|_Q + T^{\frac{1}{2}} \|h^{\frac{1}{2}}\beta' \cdot \nabla u_h\|_Q.$$

By Lemma 3.1 and Lemma 3.2 we have

$$\begin{aligned} \inf_{v_h \in V_h} \|h^{\frac{1}{2}}(\bar{\beta} \cdot \nabla u_h - v_h)\|_Q &\lesssim h^{\frac{1}{2}} \|\bar{\beta}\|_{W^{1,\infty}(\Omega)} \|u_h\|_Q + \left( \int_I \bar{s}_h(u_h, u_h) dt \right)^{\frac{1}{2}} \\ &\lesssim h^{\frac{1}{2}} \|\bar{\beta}\|_{W^{1,\infty}(\Omega)} \|u_h\|_Q + h^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \nabla u_h\|_Q + \left( \int_I s_h(u_h, u_h) dt \right)^{\frac{1}{2}} \end{aligned}$$

It follows that

$$\begin{aligned} \inf_{v_h \in V_h} \|h^{\frac{1}{2}}(\bar{\beta} \cdot \nabla u_h - v_h)\|_Q \\ \lesssim \max(h^{\frac{1}{2}} T^{\frac{1}{2}} \|\bar{\beta}\|_{W^{1,\infty}(\Omega)}, \|\beta\|_{L^{\infty}(\Omega)}^{\frac{1}{2}}) (\sup_{t \in I} \|u_h(\cdot, t)\|_{\Omega} + \|u_h\|_h). \end{aligned} \quad (5.10)$$

Using the assumption on the small scale fluctuations  $\|\beta'\|_{L^{\infty}(Q)}^2 \lesssim \mu$  we have

$$\|h^{\frac{1}{2}}\beta' \cdot \nabla u_h\|_Q \lesssim h^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \nabla u_h\|_Q \leq h^{\frac{1}{2}} \|u_h\|_h.$$

We conclude by collecting terms and applying Lemma 3.3.  $\square$

*Remark 3. (The necessity of stabilization for robustness) Note that the stability of the dual problem holds regardless of the numerical method used. The stabilization in the numerical method allows us to control the first residual in the a posteriori error estimate, by using the discrete stability estimate (3.3). If no stabilization is present there is no control of the streamline derivative making it impossible to obtain uniformity in  $\mu$ . Another observation that is worthwhile is that the above argument is valid only for high macroscopic Péclet number. One may prove that the situation improves if the domain is convex so that elliptic regularity can be used, in particular we then get an estimate that is valid also in the low Reynolds number regime, but also in this case stabilization is required to yield uniformity in  $\mu$ .*

**5.1.1. The degenerate case of unknown  $\beta'$ .** In many relevant cases  $\beta'$  may be unknown, or only statistics of it known. Then some stochastic method may be used to recover expectancy values for the solution. In this section we will consider the situation, that  $\beta'$  is simply excluded from the computation and we will show that under our assumptions on the small scale velocity fluctuations the approximation result still holds. Indeed in the high Péclet number regime the consistency error made by dropping the fine scale fluctuations of  $\beta$  is smaller than the discretization error.

Here we use an advective field  $\bar{\beta}$  that we assume is divergence free and let  $\beta$  be replaced by  $\bar{\beta}$  in (3.7).

**THEOREM 5.3.** *Let  $u$  be the solution of (1.5) and  $u_h$  the solution of (3.7), with  $\bar{\beta}$  instead of  $\beta$  for the advective field, assume that  $\text{Pe}_h \gg 1$ , then*

$$\|\tilde{e}\|_{\mathfrak{h}} \lesssim C_T \left(\frac{h}{\mathfrak{h}}\right)^{\frac{1}{2}} (1 + h^{\frac{1}{2}} + T^{\frac{1}{2}} + T) \left( \int_0^T \|f\|_{\Omega} dt + \|u_0\|_{\Omega} \right).$$

*Proof.* We proceed as in the proofs of Theorems 5.1 and 5.2

$$\begin{aligned} \|\tilde{e}\|_{\mathfrak{h}}^2 &= (e(0), \varphi(0) - \pi_h \varphi)_{\Omega} + (\partial_t e, \varphi - \pi_h \varphi)_Q \\ &\quad + \int_I \left( a(e, \varphi - \pi_h \varphi) + \bar{s}_h(u_h, \pi_h \varphi) + (\beta' \cdot \nabla u_h, \varphi)_{\Omega} dt \right). \end{aligned} \quad (5.11)$$

The only thing that differs from the previous analysis is the last term in the right hand side. Except for this term the proof proceeds as before. We will therefore here only show how this term may be bounded. After an integration by parts we have

$$(\beta' \cdot \nabla u_h, \varphi)_Q = (u_h, \beta' \cdot \nabla \varphi)_Q \leq \sup_{t \in I} \|u_h(\cdot, t)\|_{\Omega} T \|\beta'\|_{L^{\infty}(Q)} \mathfrak{h}^{-1/2} \sup_{t \in I} \|\mathfrak{h}^{\frac{1}{2}} \nabla \varphi(\cdot, t)\|_{\Omega}^2.$$

Note that under our assumptions  $\|\beta'\|_{L^{\infty}(Q)} \sim \mu^{\frac{1}{2}} \leq \|\beta\|_{L^{\infty}(Q)}^{\frac{1}{2}} h^{\frac{1}{2}}$  leading to

$$(\beta' \cdot \nabla u_h, \varphi)_Q \lesssim \left(\frac{h}{\mathfrak{h}}\right)^{\frac{1}{2}} C_T T \|\tilde{e}\|_{\mathfrak{h}}.$$

□

It follows that the consistency error is of the same order as the discretization error. If the Péclet number is large, the contribution from the discretization error can be assumed to be dominating and the same order of convergence as in the unperturbed case should be observed, until the Péclet number becomes so small that the inconsistency dominates. This means that in the high Péclet regime, if data are known to be rough, noise in the velocities satisfying the constraint on  $\beta'$  may be neglected. Hence in any stochastic analysis or computation at high Reynolds number fluctuations must be larger than  $\mu^{\frac{1}{2}}$  to be essential.

**5.2. Conclusion.** We have derived robust a posteriori and a priori error estimates for transient convection–diffusion equations. The upshot is that the estimates are completely robust with respect to the Péclet number, in the sense that we also control the Sobolev norms of the exact solution in the error constant. The estimates allow for low regularity data and multiscale advection, that may have strong spatial variation on the fine scale under a special scale separation condition. It is easy to see that a general high Péclet transport problem with very strong spatial variation in the advection velocity must be ill-posed since a small perturbation of the initial position of a flow particle may have an important influence on its trajectory. So our work is an attempt to separate computable cases from inherently unstable ones, using scale separation. In this spirit, our paper proposes a first step towards an understanding of what transport problems are computable in the high Péclet regime, beyond the standard assumption of smooth data and sets a baseline for what should be achieved theoretically in the analysis of more involved methods, such as multiscale methods, in order to claim that they produce an accuracy beyond what is obtained using a standard stabilized finite element method.

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